ON THE ENERGY OF SOUND WAVES

(OB ENERGII ZVUKOVYKH VOLN)

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1. The change in the energy of a gas moving in a gravitational field is determined by the equation

$$\frac{\partial}{\partial t}\rho\left(\frac{v^2}{2}+\varepsilon\right)+\operatorname{div}\rho\left(\frac{v^2}{2}+w\right)\mathbf{v}-\rho\mathbf{v}\mathbf{g}=0 \qquad (1.1)$$

Here t is time, $\mathbf{v}$ the velocity of the particles, $\mathbf{g}$ the acceleration of gravity, ρ the density, ε the internal specific energy, and w the specific enthalpy. The quantity w is related to ε and to the pressure p of the gas by the formula

$$\rho w = \rho\varepsilon + p$$

We will consider the propagation of a sound wave in a nonhomogeneous medium. We will assume that in the undisturbed state the pressure p_0, the density ρ_0, the internal energy ε_0, and the relative enthalpy w_0 do not change with time, and are given as functions of the coordinates. For the sake of simplicity, we will assume that the velocity is zero in the undisturbed state, i.e.

$$\mathbf{v}_0 = 0$$

The equilibrium equation of the medium in this case is

$$\operatorname{grad} p_0 = \rho_0\mathbf{g} \qquad (1.2)$$

Let us simplify Equation (1.1) by making use of the smallness of the amplitude of a sound wave. For this purpose we expand the quantity ρw into a series in terms of the thermodynamic variables of pressure p, and of relative entropy s. Let T denote the temperature of the material. In accordance with thermodynamic principles

$$dw = Tds + \frac{1}{\rho}dp$$

Making use of this equation, we can write the resulting expansion for ρw in terms of the first and second order of infinitesimals as

$$\rho w = \rho_0 w_0 + \left(1 + \frac{w_0}{a_0^2}\right) p' + \left[\rho_0 T_0 + w_0 \left(\frac{\partial \rho}{\partial s_0}\right)_p\right] s' +$$
$$+ \frac{1}{2}\left[\frac{1}{\rho_0 a_0^2} + w_0 \left(\frac{\partial^2 \rho}{\partial p_0^2}\right)_s\right] p'^2 + \left(\frac{T_0}{a_0^2} + w_0 \frac{\partial^2 \rho}{\partial p_0 \partial s_0}\right) p's' +$$
$$+ \frac{1}{2}\left[2T_0 \left(\frac{\partial \rho}{\partial s_0}\right)_p + \frac{\rho_0 T_0}{c_{p_0}} + w_0 \left(\frac{\partial^2 \rho}{\partial s_0^2}\right)_p\right] s'^2$$

Here a is the velocity of sound; c_p is the specific heat capacity at constant pressure; p' and s' are the deviations of pressure, and entropy from their equilibrium values. The subscript zero refers to quantities in the undisturbed state.

Let us substitute the last displayed expression into Equation (1.1). For the simplification of the resulting expression we make use of the equilibrium equation (1.2), and also of the continuity equations of Euler, and of the equations of the conservation of entropy, in which we drop the quadratic terms. We thus obtain the equation which expresses the law of the conservation of the energy in a sound wave

$$\frac{\partial}{\partial t} \frac{1}{2}\left(\rho_0 v^2 + \frac{1}{\rho_0 a_0^2} p'^2\right) + \operatorname{div} p'\mathbf{v} + \frac{1}{a_0^2}\left(\frac{\partial p}{\partial s_0}\right)_\rho s'\mathbf{vg} = 0 \qquad (1.3)$$

If in the equilibrium state the entropy of the medium is constant in the entire volume under consideration, i.e. if

$$s_0 = \text{const}$$

then it is easy to see that $s' = 0$. In this case the last term of the Equation (1.3) will disappear. It will also vanish when the gravity field is absent.

In the propagation of a sound wave in an ideal gas with c_p = const, the Equation (1.3) can be given the form

$$\frac{\partial}{\partial t} \frac{1}{2}\left(\rho_0 v^2 + \frac{1}{\rho_0 a_0^2} p'^2\right) + \operatorname{div} p'\mathbf{v} + \frac{\rho_0}{c_p} s'\,\mathbf{vg} = 0$$

Let us now consider the propagation of a short sound wave, i.e. of a wave with a narrow band of the disturbed flow. The amplitude and direction of such a wave hardly change over distances of the order of the width of the disturbed region λ_*. When $\lambda_* \to 0$, the following first approximate relations, determining the shock front, hold between the parameters of the gas

$$\mathbf{v} = v\mathbf{n}, \qquad v = \frac{1}{\rho_0 a_0} p', \qquad s' = 0 \qquad (1.4)$$

Here $\mathbf{n}$ is the normal to the wave front.

In a short wave, the density of the sound energy e, and the density of the stream of sound energy $\mathbf{q}$ are connected by an equation that characterizes a planar traveling impulse of small amplitude

$$\mathbf{q} = a_0 e\mathbf{n}, \qquad e = \frac{1}{2}\left(\rho_0 v^2 + \frac{1}{\rho_0 a_0^2} p'^2\right), \qquad \mathbf{q} = p'\mathbf{v} \tag{1.5}$$

Substituting (1.4) and (1.5) into the relation (1.3), we obtain the fundamental equation of geometrical acoustics

$$\frac{\partial a_0 e}{\partial t} + a_0 \mathbf{n} \operatorname{grad} a_0 e + a_0^2 e \operatorname{div} \mathbf{n} = 0$$

Let us introduce the derivative along a ray along which an element of the wave moves

$$\frac{d}{dt} = \frac{\partial}{\partial t} + a_0 (\mathbf{n}\nabla)$$

Integrating the last equation, we obtain

$$e = e_0 \frac{a_{00}}{a_0} \exp\left(-\int_{t_0}^{t} a_0 \operatorname{div} \mathbf{n}\, dt\right) \tag{1.6}$$

Here e_0 and a_{00} denote the density of the sound energy, and the equilibrium density at the initial moment of time $t = t_0$, taken at the initial point of the ray.

The Equation (1.6) determines the change in the intensity of the sound along the path of a wave element. Let p_0' and ρ_{00} denote the surplus pressure and the equilibrium density at the initial moment of time $t = t_0$. From the Formula (1.6) follows the law of the change of the amplitude of the sound wave

$$p' = p_0' \sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}} \exp\left(-\frac{1}{2}\int_{t_0}^{t} a_0 \operatorname{div} \mathbf{n}\, dt\right) \tag{1.7}$$

The Formula (1.7) has been derived in a different way by Keller [1], where it is transformed into

$$p' = p_0' \sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}} \sqrt{\frac{f_0}{f}} \tag{1.8}$$

Here f is the area of a cross section of the elementary ray tube at the time moment t, f_0 is the cross section of the tube when $t = t_0$.

2. Let us now consider the motion of a short wave of small amplitude in the approximation of geometrical acoustics. In this approximation the velocity of the shock wave is different from the velocity of propagation of sound waves, while its amplitude decays according to a different law

than (1.8). For the sake of simplicity let us assume that all surplus quantities in a sound impulse with a shock wave have a triangular profile. Let p_*' denote the amplitude of the shock wave in the approximation of geometric acoustics; let λ_0 be the initial length of the sound impulse, and V be the specific volume of the gas equal to $1/\rho$. We introduce the coefficient

$$m_0 = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)_s$$

which is equal to $(\kappa + 1)/2$, where κ is the adiabatic index of Poisson. Making use of the obtained results, one can show that the amplitude p_*' of the shock wave is determined by the Formula [2-4]

$$p_*' = p_0'\sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}}\sqrt{\frac{f_0}{f}}\left(1 + \frac{p_0'}{\lambda_0}\sqrt{\frac{a_{00}}{\rho_{00}}}\int_{l_0}^{l}\frac{m_0}{\sqrt{\rho_0 a_0^5}}\sqrt{\frac{f_0}{f}}\,dl\right)^{-1/2} \tag{2.1}$$

Here dl is the ray's element of length, which is equal to $a_0 dt$.

The length λ_* of the wave changes in accordance with the Equation [3,4]

$$\lambda_* = \lambda_0 \frac{a_0}{a_{00}}\left(1 + \frac{p_0'}{\lambda_0}\sqrt{\frac{a_{00}}{\rho_{00}}}\int_{l_0}^{l}\frac{m_0}{\sqrt{\rho_0 a_0^5}}\sqrt{\frac{f_0}{f}}\,dl\right)^{1/2} \tag{2.2}$$

Let us compute the total energy E per unit of time of an elementary sound impulse contained within a ray tube with cross section area f. The total energy of a sound impulse, with a triangular profile for the surplus pressure, whose amplitude and length are given by the Formulas (2.1) and (2.2), respectively, is equal to

$$E = \frac{1}{3}\frac{\lambda_0 f_0 p_0'}{\sqrt{\rho_{00} a_{00}^3}}\frac{1}{\sqrt{\rho_0 a_0}}\sqrt{\frac{f}{f_0}}\,p_*'$$

The change of energy of the considered impulse is determined by the derivative*

$$\frac{dE}{dt} = \frac{1}{3}\frac{\lambda_0 f_0 p_0'}{\sqrt{\rho_{00} a_{00}^3}}\, a_0 \frac{d}{dl}\left(\frac{1}{\sqrt{\rho_0 a_0}}\sqrt{\frac{f}{f_0}}\,p_*'\right)$$

Making use of the Equation (2.1), we find

$$\frac{d}{dl}\left(\frac{1}{\sqrt{\rho_0 a_0}}\sqrt{\frac{f}{f_0}}\,p_*'\right) = -\frac{1}{2}\frac{\sqrt{\rho_{00} a_{00}^3}}{\lambda_0 p_0'}\frac{m_0}{\rho_0^2 a_0^4}\frac{f}{f_0}\,p_*'^3$$

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* We note that in the approximations of geometrical acoustics $E = \lambda_0 f_0 e_0/3$ and $dE/dt = 0$.

This yields an expression for the sought change of the total energy of an elementary sound impulse

$$\frac{dE}{dt} = -\frac{1}{6}\frac{m_0}{\rho_0^2 a_0^3} f p_*'^3 \tag{2.3}$$

We will show that the found change in the quantity E is due to the dissipation of the energy of the shock wave. The value of this dissipation, caused by the viscosity and heat-conductivity, can be computed for an ideal fluid on the basis of the same change in entropy s_*' which occurs in a shock wave. The quantity s_*' is an infinitesimal of the third order in the surplus pressure p_*', and is determined by the formula

$$s_*' = \frac{1}{6}\frac{m_0}{\rho_0^3 a_0^4 T_0} p_*'^3$$

With the aid of the derived expression, we can find the energy which in a unit of time is scattered in the form of heat on an element of the shock front of area f. Let Q denote the dissipated energy. Its change per unit time on the separated element of the shock front is

$$\frac{dQ}{dt} = \frac{1}{6}\frac{m_0}{\rho_0^2 a_0^3} f p_*'^3$$

which coincides with the Expression (2.3) taken with the opposite sign. This fact makes it possible to find (without the computation of the approximation of geometric acoustics) the law of the decay of a small amplitude shock wave moving in an inhomogeneous medium. The change in energy of an elementary sound impulse of length λ_*, contained in a ray tube of cross section area f, and having on the front the surplus pressure p_*', will be

$$\frac{d}{dt}\left(\frac{1}{\rho_0 a_0^2} f \lambda_* p_*'^2\right) = -\frac{1}{2}\frac{m_0}{\rho_0^2 a_0^3} f p_*'^3 \tag{2.4}$$

Here the quantity λ_* satisfies the relation [4]

$$\frac{d\lambda_*}{dt} = \frac{\lambda_*}{a_0}\frac{da_0}{dt} + \frac{1}{2} m_0 \frac{p_*'}{\rho_0 a_0} \tag{2.5}$$

The system of ordinary differential equations (2.4) and (2.5) must be solved with the initial condition

$$\lambda_* = \lambda_0 \qquad p_*' = p_0' \quad \text{when } t = t_0$$

It is not difficult to prove that the solution of the system of Equations (2.4) and (2.5), that satisfies this condition, is given by the Formulas (2.1) and (2.2).

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